Localized X-shaped field generated by a superluminal electric charge

Erasmo Recami,[1,2,*] Michel Zamboni-Rached,[3] and César A. Dartora[3]
[1]*Facoltà di Ingegneria, Università statale di Bergamo, Dalmine (BG), Italy*
[2]*INFN—Sezione di Milano, Milan, Italy*
[3]*DMO, FEEC, State University at Campinas, Campinas, São Paulo, Brazil*

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It is now well known that Maxwell equations admit of wavelet-type solutions endowed with arbitrary group velocities ($0<v_{\rm g}<\infty$). Some of them, which are rigidly moving and have been called localized solutions, attracted large attention. In particular, much work has been done with regard to the superluminal localized solutions (SLSs), the most interesting of which are the "X-shaped" ones. The SLSs have been actually produced in a number of experiments, always by suitable interference of ordinary-speed waves. In this paper we show, by contrast, that even a superluminal charge creates an electromagnetic X-shaped wave: namely, on the basis of Maxwell equations, we are able to evaluate the field associated with a superluminal charge (under the approximation of pointlikeness): It results in constituting a very simple example of a *true* X wave.

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I. INTRODUCTION

It is well known that Maxwell equations have been shown to admit of wavelet-type solutions endowed with arbitrary [1] group velocities $0<v_{\rm g}<\infty$. Some of them, which are rigidly moving and have been called "localized solutions," attracted much attention [2]. In particular, much work has been done with regard to the superluminal localized solutions (SLS's), the most interesting of which—as predicted by special relativity (SR) itself [3]—are the "X-shaped" ones [4]. Such X-shaped SLSs have been actually produced in a number of experiments [5].

The theory of SR, when based on the *ordinary* postulates but not restricted to subluminal waves and objects, i.e., in its extended version [6], predicts the simplest X-shaped wave to be the one corresponding to the electromagnetic field created by a superluminal[1] charge [8,9]. Evaluating the field associated with a superluminal electric charge is of utmost importance not only as a contribution to the theory of the X-shaped waves, but also as a starting point for studying the electromagnetic interaction of a charged "tachyon" with ordinary matter (and planning, may be, the construction of a suitable detector).

II. THE TOY MODEL OF A POINTLIKE SUPERLUMINAL CHARGE

Let us first start by considering, formally, a pointlike superluminal charge, even if the hypothesis of pointlikeness (already unacceptable in the subluminal case) is totally inadequate in the superluminal case, as it was thoroughly shown in Refs. [8].

Then, let us consider the ordinary vector potential A^μ and a current density $j^\mu \equiv (0,0,j_z;j^0)$ flowing in the z direction. On assuming the fields to be generated by the sources only, one has that $A^\mu \equiv (0,0,A_z;\phi)$, which, when adopting the Lorentz gauge, obeys the equation $\Box A^\mu = 4\pi j^\mu/c$. Such a nonhomogeneous wave equation, in cylindrical coordinates $(\rho,\theta,z;t)$ and for axial symmetry [which requires *a priori* that $A^\mu = A^\mu(\rho,z;t)$], can be written as[2]

$$\left[-\rho\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right)+\frac{1}{\gamma^2}\frac{\partial^2}{\partial\zeta^2}+\frac{1}{\gamma^2}\frac{\partial^2}{\partial\eta^2}-4\frac{\partial^2}{\partial\zeta\,\partial\eta}\right]A^\mu(\rho,\zeta,\eta)=\frac{4\pi}{c}j^\mu(\rho,\zeta,\eta), \tag{1}$$

provided that[3] we go on to the new "V-cone variables" [11], with $V^2>c^2$:

$$\zeta \equiv z - Vt,$$
$$\eta \equiv z + Vt. \tag{2}$$

In Eq. (1), it is $A^\mu \equiv (0,0,A_z;\phi)$; quantity μ assumes the two values $\mu=3,0$ only; and [6]

$$\gamma^2 = \frac{1}{V^2-1}. \tag{1'}$$

Let us now suppose A^μ to be independent of η, so that $A^\mu = A^\mu(\rho,\zeta)$. Due to Eq. (1), we shall have $j^\mu = j^\mu(\rho,\zeta)$ too; and therefore $j_z = Vj^0$ (from the continuity equation) and

*Email address: recami@mi.infn.it

[1]Incidentally, let us recall that the *luminal* case was successfully examined by Bonnor [7], who showed the Maxwell equations to admit of finite-energy solutions even in the limiting case of a (mass-free) "particle" carrying equal amounts of positive and negative electric charge.

[2]As a further check of our calculations, we started also from the so-called scalar Bromwich-Borgnis [10] potential u, under the hypothesis that $\boldsymbol{j}=(0,0,j_z)$, in which case it is $E_\rho = \partial^2 u/\partial\rho,\partial z$; while $E_z = -\partial^2 u/\partial\tau^2 + \partial^2 u/\partial z^2$; and $B_\phi = \partial^2 u/\partial\rho\,\partial\tau$, where $\tau = ct$. On defining the function $\psi \equiv A_z \equiv \partial u/\partial\tau$, we showed by Maxwell equations that ψ has to obey the same nonhomogeneous (axially symmetric) wave equation (1), with $\mu=3$.

[3]In the following we shall put $c=1$, whenever convenient.

$A_z = V\phi/c$ (from the Lorentz gauge). Then, by calling $\psi \equiv A_z$, so that $\phi = c\psi/V$, Eq. (1) yields the hyperbolic equation

$$\left[-\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \frac{1}{\gamma^2}\frac{\partial^2}{\partial\zeta^2}\right]\psi(\rho,\zeta) = 4\pi j_z(\rho,\zeta). \quad (3)$$

One can notice that the procedure leading to Eq. (3) constitutes a simple *generalization* of the theorem by Lu *et al.* [12] to nonhomogeneous equations, i.e., to the case with sources [13].

Let us finally analyze the possibility and consequences of having a superluminal pointlike charge e, traveling with constant speed V along the z axis $(\rho=0)$ in the positive direction:

$$j_z = eV\frac{\delta(\rho)}{\rho}\delta(\zeta). \quad (4)$$

To solve Eq. (3) with j_z given by Eq. (4), let us apply (with respect to the variable ρ) the Fourier-Bessel (FB) transformation of a function $f(x)$ into function $F(\Omega)$: $f(x) = \int_0^\infty \Omega F(\Omega) J_0(\Omega x) d\Omega$, and $F(\Omega) = \int_0^\infty x f(x) J_0(\Omega x) \mathrm{d}x$, quantity $J_0(\Omega x)$ being the ordinary zero-order Bessel function. After some calculations, one gets the equation

$$\left[\frac{1}{\gamma^2}\frac{\partial^2}{\partial\zeta^2} + \Omega^2\right]\Psi(\Omega,\zeta) = 4\pi eV\delta(\zeta). \quad (5)$$

By applying subsequently the ordinary Fourier transformation with respect to the variable ζ (going on, from ζ, to the variable ω), after some further manipulations we obtain

$$\Psi(\Omega,\omega) = 4\pi eV\frac{\gamma^2}{\gamma^2\Omega^2 - \omega^2}. \quad (6)$$

Finally, the solution of our equation is got by performing the corresponding *inverse* Fourier and FB transformations:

$$\psi(\rho,\zeta) = \sqrt{8\pi}eV\gamma^2\int_{-\infty}^{\infty}\mathrm{d}\omega\int_0^\infty \mathrm{d}\Omega\,\frac{\Omega J_0(\Omega\rho)e^{-i\omega\zeta}}{\gamma^2\Omega^2-\omega^2}, \quad (7)$$

which, on using formulas (3.723.9) and (6.671.7) of Ref. [14], yields

$$\psi(\rho,\zeta) = 0 \quad \text{for } 0 < \gamma|\zeta| < \rho,$$
$$\psi(\rho,\zeta) = \sqrt{8\pi}e\frac{V}{\sqrt{\zeta^2-\rho^2(V^2-1)}} \quad \text{for } 0 \leqslant \rho < \gamma|\zeta|. \quad (8)$$

In Fig. 1 we show such a solution $A_z \equiv \psi$ as a function of ρ and ζ, evaluated for $\gamma=1$ (i.e., for $V = c\sqrt{2}$).

For comparison, one may recall that the *classical* X-shaped solution [4] of the *homogeneous* wave equation has the form [11] (with $a>0$)

$$X = \frac{V}{\sqrt{(a-i\zeta)^2+\rho^2(V^2-1)}}. \quad (9)$$

In the second one of Eqs. (8) it enters expression (9) with the spectral parameter [11] $a=0$, which indeed corresponds to

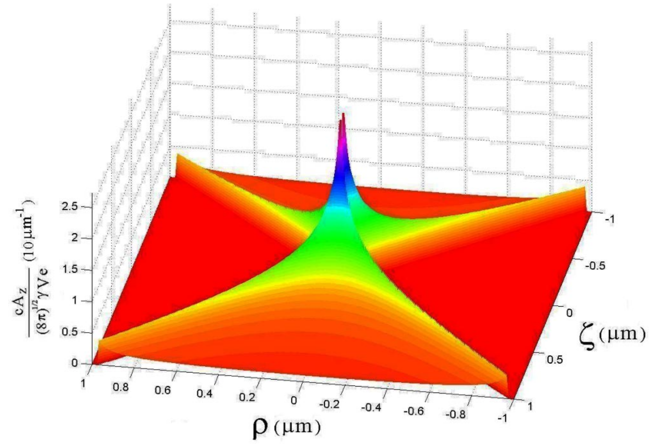

FIG. 1. Behavior of $A_z \equiv \psi$ as a function of ρ and of $\zeta \equiv z - Vt$ evaluated for $\gamma = 1$ (i.e., for $V = c\sqrt{2}$). (Of course, we skipped the points in which A_z must diverge, namely, the vertex and the cone surface).

the nonhomogeneous case (the fact that for $a=0$ these equations differ for an imaginary unit will be discussed elsewhere).

It is rather important, at this point, to notice that such a solution, Eq. (11), represents a wave existing only inside the (unlimited) double cone $\mathcal{C}$ generated by the rotation around the z axis of the straight lines $\rho = \pm\gamma\zeta$: This is in full agreement with the predictions [15] of the "extended" theory of special relativity [6].

III. EVALUATING THE FIELDS GENERATED BY THE SUPERLUMINAL CHARGE

Once solution (8) for the "potential" ψ has been found, we can evaluate the corresponding electromagnetic fields. The standard relations $\boldsymbol{E} = -\boldsymbol{\nabla}\phi - \partial\boldsymbol{A}/\partial t$ and $\boldsymbol{H} = \boldsymbol{\nabla}\times\boldsymbol{A}$ imply in the present case $[\psi = \psi(\rho,\zeta) \equiv A_z$; and $\phi = c\,\psi/V]$ that, when $0 \leqslant \rho < \gamma|\zeta|$ (i.e., inside the cone $\mathcal{C}$), the fields become[4]

$$E_\rho = -\sqrt{8\pi}e\rho\frac{V^2-1}{\sqrt{[\zeta^2-\rho^2(V^2-1)]^3}}, \quad (10a)$$

$$E_z = -\sqrt{8\pi}e\zeta\frac{V^2-1}{\sqrt{[\zeta^2-\rho^2(V^2-1)]^3}}, \quad (10b)$$

$$H_\phi = -\sqrt{8\pi}e\rho\frac{V(V^2-1)}{\sqrt{[\zeta^2-\rho^2(V^2-1)]^3}}, \quad (10c)$$

where, let us recall, $\zeta \equiv z - Vt$, with $V^2 > c^2$. We show in Fig. 2 the direction of the various field components in our coor-

[4]It should be noted that the same results are obtained when starting from the four-potential associated with a subluminal charge (e.g., an electric charge at rest), and then applying to it the suitable superluminal lorentz "transformation" [6].

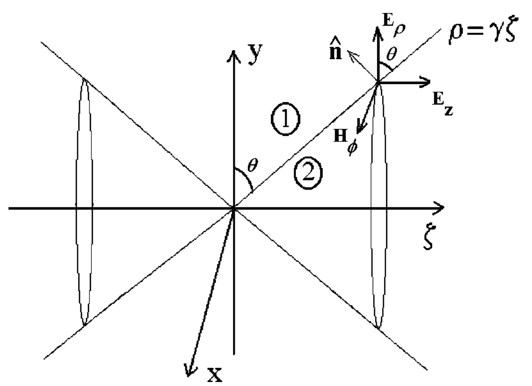

FIG. 2. Depicted here is the direction of the various field components, in our coordinates.

dinates; while the behavior of E_z, as a function of ρ and ζ, is shown in Fig. 3.

However, outside the cone $\mathcal{C}$, i.e., for $0<\gamma|\zeta|<\rho$, one gets, as expected, that

$$E_\rho = E_z = H_\phi = 0. \tag{10d}$$

One faces therefore a field discontinuity when crossing the double-cone surface, since the field is zero outside it. Nevertheless, the boundary conditions imposed by Maxwell equations [15] are satisfied by our solution (8) or Eqs. (10a)–(10c), since at each point of the cone surface the electric and the magnetic field are both tangent to the cone: We shall discuss this point below.

Let us here emphasize that, when $V\rightarrow\infty, \gamma\rightarrow 0$, the electric field tends to vanish, while the magnetic field tends to the value $H_\phi = -\sqrt{8\pi}e/\rho^2$. This does agree with what is expected from extended relativity [9], which predicts superluminal charges to behave, in a sense, as magnetic monopoles. In the present paper we can only mention such a circumstance, and refer to Refs. [3,9], where it is shown that, if

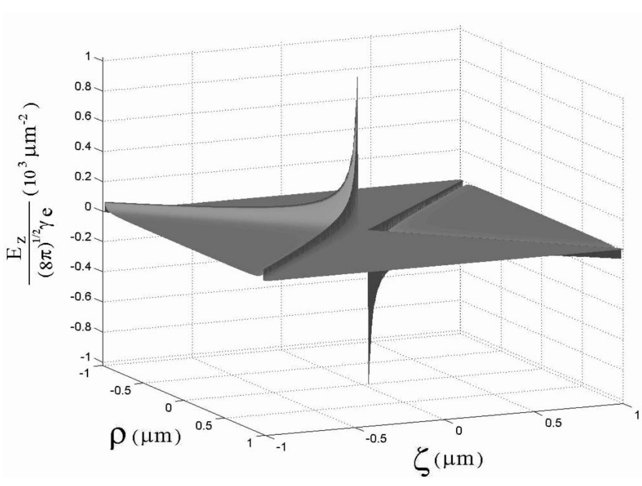

FIG. 3. Behavior of the z component of the electric field generated by a superluminal (pointlike) charge as a function of ρ and ζ, with the same parameters as used for Fig. 1. (Once again, we skipped the points in which E_z has to diverge, namely, the vertex and the cone surface).

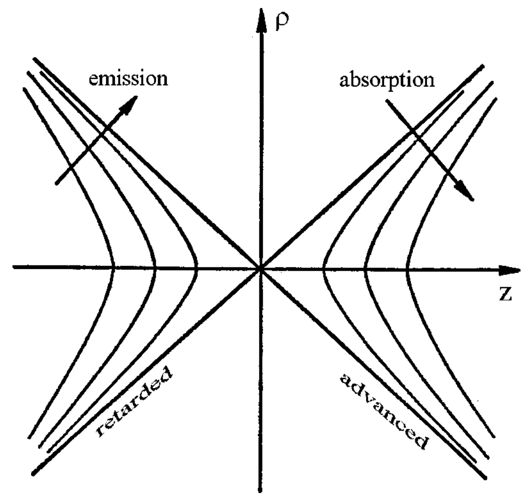

FIG. 4. The spherical equipotential surfaces of the electrostatic field created by a charge at rest get transformed into two-sheeted rotation hyperboloids, contained inside an unlimited double cone, when the charge travels at superluminal speed (cf. Refs. [6,8]). This figure shows, among others, that a superluminal charge traveling at constant speed, in a homogeneous medium such as the vacuum, does *not* lose energy [8]. Let us mention, incidentally, that this double cone has nothing to do with the Cherenkov cone (see the text). The present picture is a reproduction of Fig. 27 of our earlier work [6].

one calls *electric* the ''electromagnetic charge'' when it is subluminal, then he should call it *magnetic* when superluminal[5] (cf. Fig. 46 at page 155 of Ref. [6(a)]). Actually, result (8) can be obtained in a quicker way just by applying a superluminal lorentz ''transformation'' [6] to the fields generated by a subluminal (in particular, at rest) electric point charge.

Let us add that—as mentioned at the end of the preceding section—extended relativity predicts, e.g., that the spherical equipotential surfaces of the electrostatic field created by a charge at rest get transformed (by a superluminal lorentz transformation) into two-sheeted rotation hyperboloids, contained inside an unlimited double cone [6,8]: see Fig. 4. One ought to notice, incidentally, that this double cone does not have much to do with the Cherenkov cone: In fact, the double cone is associated with a constant-speed superluminal charge even in the vacuum, while Cherenkov radiation emission is induced by a fast electric charge only out of a material medium. Moreover (cf. also Fig. 27 at page 80 of Ref. [6(a)]) a superluminal charge traveling at constant speed, in the vacuum, e.g., does *not* lose energy [8].

Let us go eventually back to the problem where one faces a field discontinuity across the double-cone surface [see Eqs. (10a)–(10c) and Eqs. (10d)], since the field is zero outside $\mathcal{C}$;

[5]We have shown elsewhere [9,16] that a superluminal charge e and a superluminal current j^μ are pseudoscalar and pseudovector, respectively: Just as in the case of a magnetic charge and a magnetic current; so that they should rather be written as $\gamma_5 e$ and $\gamma_5 j^\mu$. But in this paper we shall forget about the symmetry properties of those quantities.

for $\rho \to \gamma|\zeta|$ fields (10a)–(10c) even diverge. Nevertheless, one can straightforwardly *verify* that our solution (8), or Eqs. (10a)–(10d), satisfies the following boundary conditions, required by Maxwell equations in the present case of a *moving* boundary [17,18]:

$$(\boldsymbol{E}_{\rm ext}-\boldsymbol{E}_{\rm int})\cdot\hat{\boldsymbol{n}}=\sigma,$$
$$(\boldsymbol{H}_{\rm ext}-\boldsymbol{H}_{\rm int})\cdot\hat{\boldsymbol{n}}=0,$$
$$(\boldsymbol{E}_{\rm ext}-\boldsymbol{E}_{\rm int})_{\tan}=-(\hat{\boldsymbol{n}}\cdot\boldsymbol{V})\hat{\boldsymbol{n}}\times(\boldsymbol{H}_{\rm ext}-\boldsymbol{H}_{\rm int}),$$
$$[1-(\hat{\boldsymbol{n}}\cdot\boldsymbol{V})^2]\hat{\boldsymbol{n}}\times(\boldsymbol{H}_{\rm ext}-\boldsymbol{H}_{\rm int})=\boldsymbol{j}. \qquad (11)$$

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